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The composite operator $T\bar{T}$ in sinh-Gordon and a series of massive minimal models

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Abstract: The composite operator $T\bar{T}$, obtained from the components of the energy-momentum tensor, enjoys a quite general characterization in two-dimensional quantum field theory also away from criticality. We use the form factor bootstrap supplemented by asymptotic conditions to determine its matrix elements in the sinh-Gordon model. The results extend to the breather sector of the sine-Gordon model and to the minimal models $\mathcal{M}_{2/(2N+3)}$ perturbed by the operator $\phi_{1,3}$.

Keywords: Exact S-Matrix, Integrable Field Theories, Field Theories in Lower Dimensions.

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Contents

1. Introduction

The ability of a quantum field theory to describe a system with infinitely many degrees of freedom is reflected by an infinite-dimensional operator space. In two dimensions, the detailed structure of the operator space at a generic fixed point of the renormalization group was revealed by the solution of conformal field theories [1]. It is divided into different operator families, each one consisting of a primary and infinitely many descendants. Within an operator family, the scaling dimensions differ from that of the primary by integer numbers that label different 'levels'.

Perturbative arguments lead to the conclusion that this same structure is maintained when conformal invariance is broken by a perturbation producing a mass scale [2]. If the massive theory is integrable, the operator space can be studied non-perturbatively within the form factor bootstrap approach [3, 4]. It was shown for several models that the global counting of solutions of the form factor equations matches that expected from conformal field theory [5].

As for the correspondence between solutions of the form factor equations and operators, asymptotic conditions at high energies play a crucial role. While primary operators are naturally associated to the solutions with the mildest asymptotic behavior, we argued

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in [6] that specific asymptotic conditions selecting the solutions according to the level can be identified. These were used in [7] to show the isomorphism between the critical and off-critical operator spaces in the Lee-Yang model, level by level up to level 7.

Asymptotic conditions, however, cannot determine completely a descendant operator in a massive theory. Indeed, they leave unconstrained terms which are subleading at high energies and depend on the way the operator is defined away from criticality. The operator $T\bar{T}$, obtained from the components of the energy-momentum tensor, appears as the natural starting point in relation to the problem of the off-critical continuation of descendant operators. Indeed, being the lowest non-trivial scalar descendant of the identity, this operator allows for a quite general characterization in two-dimensional quantum field theory. A. Zamolodchikov showed how to define it away from criticality subtracting the divergences which arise in the operator product expansion of T and $\bar{T}$ [8]. We showed in [6] for the massive Lee-Yang model that, with this information, the form factor programme outlined above allows to uniquely determine $T\bar{T}$ up to an additive derivative ambiguity which is intrinsic to this operator. Our results have been successfully compared with conformal perturbation theory in [9].

In this paper we address the problem of determining $T\bar{T}$ in the sinh-Gordon model. The essential difference with respect to the Lee-Yang case is that, while the latter is a minimal model with the smallest operator content (two operator families), the sinh-Gordon model possesses a continuous spectrum of primary operators, a circumstance that seriously complicates the identification of specific solutions of the form factor equations. The massive Lee-Yang model is the first in the infinite series of the $\phi_{1,3}$-perturbed minimal models $\mathcal{M}_{2/(2N+3)}$, each one containing $N+1$ operator families. Due to a well known reduction mechanism [10 – 13], these massive minimal models have to be recovered from sinh-Gordon under analytic continuation to specific imaginary values of the coupling. The form factor solution for the operator $T\bar{T}$ of the sinh-Gordon model that we construct satisfies this requirement.

The paper is organized as follows. In the next section 2 we recall a number of facts about bosonic theories in two dimensions. The form factor solutions for the primary operators in the sinh-Gordon model are reviewed in section 3, while the solution for $T\bar{T}$ is constructed in section 4. Few final remarks are contained in section 5. Six appendices conclude the paper.

2. Bosonic field with a charge at infinity

If $B_{\lambda\mu\nu}$ is a tensor antisymmetric in the first two indices, the energy-momentum tensor $T_{\mu\nu}$ of a quantum field theory can be modified into

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} + \partial^\lambda B_{\lambda\mu\nu} \tag{2.1}$$

preserving the conservation $\partial^\mu \tilde{T}_{\mu\nu} = 0$ and the total energy-momentum $P_\nu = \int d\sigma^\mu T_{\mu\nu}$. For a neutral two-dimensional boson with action

$$\mathcal{A} = \int d^2x \left[\frac{1}{2}\left(\partial_\mu\varphi\right)^2 + \mathcal{V}(\varphi)\right] \tag{2.2}$$

the choice[1]

$$B_{\lambda\mu\nu} = -\frac{iQ}{\sqrt{2\pi}}\epsilon_{\lambda\mu}\epsilon_{\nu\rho}\partial^{\rho}\varphi \tag{2.3}$$

leads to

$$\partial^{\lambda}B_{\lambda\mu\nu} = -\frac{iQ}{\sqrt{2\pi}}(\partial_{\mu}\partial_{\nu} - \eta_{\mu\nu}\Box)\varphi \tag{2.4}$$

and to the variation

$$\tilde{\Theta} = \Theta + \frac{iQ}{2}\sqrt{\frac{\pi}{2}}\,\Box\varphi \tag{2.5}$$

in the trace of the energy-momentum tensor

$$\Theta = \frac{\pi}{2}\,T^{\mu}_{\mu}\ . \tag{2.6}$$

The canonical definition $T^{\mu}_{\mu} = -2\mathcal{V}$ and the equation of motion $\Box\varphi = \partial\mathcal{V}/\partial\varphi$ give the classical result

$$\tilde{\Theta}_{cl} = \pi\left(-1 + \frac{iQ}{\sqrt{8\pi}}\frac{\partial}{\partial\varphi}\right)\mathcal{V}(\varphi)\ . \tag{2.7}$$

The parameter Q is dimensionless and goes under the name of "background charge" or "charge at infinity". It does not change the particle dynamics[2] but, inducing a modification of the energy-momentum tensor, essentially affects the scaling properties of the theory.

Free massless case. In the free massless case corresponding to the action

$$\mathcal{A}_0 = \frac{1}{2}\int d^2x\,(\partial_{\mu}\varphi)^2 \tag{2.8}$$

a nonvanishing Q leaves the energy-momentum tensor traceless ($\tilde{\Theta}_{cl} = \tilde{\Theta} = \mathcal{V} = 0$) and the theory conformally invariant. The central charge is

$$C = 1 - 6Q^2 \tag{2.9}$$

and the scaling dimension of the primary operators

$$V_{\alpha}(x) = e^{i\sqrt{8\pi}\,\alpha\,\varphi(x)} \tag{2.10}$$

is

$$X_{\alpha} = 2\alpha(\alpha - Q)\,. \tag{2.11}$$

A derivation of these results within the formalism of this paper is given in appendix A.

Minimal models. For real values of the background charge the bosonic model can be used to reproduce the minimal models of conformal field theory with central charge smaller than 1 [14]. Indeed, the requirement that the $2k$-point conformal correlator of an operator $V_{\alpha} \sim V_{Q-\alpha}$ is nonvanishing for any positive k selects the values

$$\alpha = \alpha_{m,n} = \frac{1}{2}\left[(1-m)\alpha_{+} + (1-n)\alpha_{-}\right], \qquad m,n = 1,2,\ldots \tag{2.12}$$

[1]We denote by $\eta_{\mu\nu}$ the flat metric tensor and by $\epsilon_{\lambda\mu}$ the unit antisymmetric tensor in two dimensions.

[2]In particular, Q does not enter the perturbative calculations based on the Lagrangian.

with

$$\alpha_{\pm} = \frac{Q \pm \sqrt{Q^2+4}}{2} \ . \tag{2.13}$$

Equations (2.9) and (2.11) then reproduce the central charge

$$C_{p/p'} = 1 - 6\,\frac{(p-p')^2}{pp'} \tag{2.14}$$

and the scaling dimensions

$$X_{m,n} = \frac{(p'm-pn)^2-(p-p')^2}{2pp'} \tag{2.15}$$

of the primary operators $\phi_{m,n}$ in the minimal models $\mathcal{M}_{p/p'}$ [1] through the identification

$$\alpha_{\pm} = \pm\left(\frac{p'}{p}\right)^{\pm\frac{1}{2}} \ . \tag{2.16}$$

Since $\alpha_+\alpha_- = -1$, one obtains the correspondence

$$\phi_{m,n} \sim V_{\alpha_{m,n}} = \exp\left\{\frac{i}{2}\left[(m-1)\frac{1}{\alpha_-} - (n-1)\alpha_-\right]\varphi\right\} \ . \tag{2.17}$$

It is well known that, although genuine minimal models (i.e. those possessing a finite number of conformal families which form an operator space closed under operator product expansion) correspond to rational values of p/p', the above formulæ in fact apply to the degenerate operators of conformal field theory for continuous values of central charge smaller than 1.

Liouville theory. A deformation of (2.8) which does not introduce any dimensional parameter is obtained adding an operator which is marginal in the renormalization group sense, namely has $X_\alpha = 2$. This requirement selects $V_{\alpha_-} \sim V_{Q-\alpha_-} = V_{\alpha_+}$. To be definite we take

$$\mathcal{A}_L = \int d^2x\,\left[\frac{1}{2}\,(\partial_\nu\varphi)^2 + \mu\, e^{\sqrt{8\pi}\,b\,\varphi}\right] , \tag{2.18}$$

where we defined

$$b = i\alpha_- \tag{2.19}$$

and μ is a coupling constant. For real values of b this is the action of Liouville field theory, which is conformal and has been extensively studied in the literature (see e.g. [15] for a list of references). Notice that the condition $\tilde{\Theta}_{cl} = 0$ gives the value $Q_{cl} = -i/b$, which coincides with the exact result

$$Q = \alpha_+ + \alpha_- = -i\left(b + \frac{1}{b}\right) \tag{2.20}$$

in the classical limit $b \to 0$. The central charge and scaling dimensions of exponential operators in Liouville field theory are given by (2.9) and (2.11) with Q given by (2.20).

Sinh-Gordon model. The sinh-Gordon model is defined by the action

$$\mathcal{A}_{shG} = \int d^2x \, \left[\frac{1}{2} \, (\partial_\nu \varphi)^2 + \mu \, e^{\sqrt{8\pi}\, b\, \varphi} + \mu' \, e^{-\sqrt{8\pi}\, b\, \varphi} \right] , \tag{2.21}$$

which can be regarded as a perturbed conformal field theory in two different ways.

The first one consists in seeing it as a deformation of the Gaussian fixed point, i.e. the conformal theory with $C = 1$. This amounts to setting $Q = 0$ keeping b as a free parameter. In such a case, both the exponentials appearing in the action have scaling dimension $-2b^2$ and are never marginal for real values of b. They play a symmetric role and the theory is invariant[3] under the transformation $\varphi \to -\varphi$. The trace of the energy-momentum tensor, being proportional to the operator which breaks conformal invariance, is $\tilde{\Theta} = \Theta \sim \mu \cosh \sqrt{8\pi} b \varphi$.

The second point of view consists in looking at (2.21) as the perturbation of the Liouville conformal theory (2.18) by the operator $e^{-\sqrt{8\pi}\, b\, \varphi}$ with scaling dimension $X_{ib} = -2(2b^2 + 1)$. Since this is now the operator which breaks conformal invariance, we have $\tilde{\Theta} \sim \mu' \, e^{-\sqrt{8\pi}\, b\, \varphi}$, a result which agrees with the classical expectation (2.7) once one uses Q_{cl} for Q.

Sine-Gordon model and its reductions. The sine-Gordon action

$$\mathcal{A}_{sG} = \int d^2x \, \left[\frac{1}{2} \, (\partial_\nu \varphi)^2 - 2\mu \, \cos \sqrt{8\pi} \, \beta \, \varphi \right] \tag{2.22}$$

can be obtained from (2.21) taking $\mu = \mu'$ and

$$\beta = -ib \, . \tag{2.23}$$

For real values of β the only direct interpretation of this action as a perturbed conformal field theory is as a deformation of the $C = 1$ conformal theory through the operators V_β and $V_{-\beta}$ with scaling dimension $2\beta^2$. The perturbation is relevant and the theory is massive for $\beta^2 < 1$. In this range the sine-Gordon model is known to be integrable and its factorized S-matrix is known exactly [16]. The particle spectrum consists of the soliton A and antisoliton $\bar{A}$ and, in the attractive range $0 < \beta^2 < 1/2$, of their neutral bound states, the breathers B_n with

$$1 \leq n < \mathrm{Int} \left(\frac{\pi}{\xi} \right) \tag{2.24}$$

and masses

$$m_n = 2M \, \sin \frac{n\xi}{2} \, ; \tag{2.25}$$

here $\mathrm{Int}(x)$ denotes the integer part of x, M is the mass of the soliton and

$$\xi = \frac{\pi \beta^2}{1 - \beta^2} \, . \tag{2.26}$$

[3] In this case the couplings μ and μ' have the same dimension and can be made equal shifting the field.

Taking $Q \neq 0$ and looking at the sine-Gordon model as a perturbation of the conformal theory (2.18) is problematic because the action (2.18) becomes complex when b is imaginary. Formally, however, this point of view leads, through the identity

$$\beta = \alpha_- = -\sqrt{\frac{p}{p'}}\,, \tag{2.27}$$

to the conformal field theories with central charge (2.14) perturbed by the operator $V_{-\beta} \sim \phi_{1,3}$, namely to the action

$$\mathcal{A}_{\mathcal{M}_{p/p'}} + \lambda \int d^2x\, \phi_{1,3}(x)\ . \tag{2.28}$$

This $\phi_{1,3}$-perturbation of the $C < 1$ conformal field theories is known to be integrable for any value of λ [2] and massive for a suitable choice of the sign of λ [17]. This choice is implied in (2.28).

The relation between the sine-Gordon model and the action (2.28) suggested by these formal reasonings can be confirmed and put on firmer grounds within a framework known as quantum group reduction [10 – 13]. This relies on the fact that the sine-Gordon S-matrix commutes with the generators of the affine quantum group $SL(2)_q$ with $q = \exp(i\pi/\beta^2)$, and that for rational values of $\beta^2 = p/p'$ a restriction can be operated in the space of particle states and operators of the model which is consistent with this algebraic structure and preserves locality. The quantum field theories obtained through this reduction mechanism indeed coincide with the perturbed minimal models (2.28).

While soliton and antisoliton transform as a doublet under the action of the quantum group, the breathers are scalars. This is why the reduction takes its simplest form when the space of states can be restricted to the breather sector. Let us recall that the amplitude for the scattering between the breathers B_m and B_n in the sine-Gordon model is[4] [16]

$$S_{mn}(\theta) = t_{(m+n)\frac{\xi}{2\pi}}(\theta) t_{|m-n|\frac{\xi}{2\pi}}(\theta) \prod_{j=1}^{\min(m,n)-1} t^2_{(|m-n|+2j)\frac{\xi}{2\pi}}(\theta)\,, \tag{2.29}$$

where

$$t_\alpha(\theta) = \frac{\tanh\frac{1}{2}(\theta + i\pi\alpha)}{\tanh\frac{1}{2}(\theta - i\pi\alpha)}\ . \tag{2.30}$$

While the double poles are associated to multiscattering processes [18, 19], the simple poles located at $\theta = i(m+n)\xi/2$ and $\theta = i(\pi - |m-n|\xi/2)$ correspond to the bound states B_{n+m} and $B_{|m-n|}$, respectively, propagating in the scattering channel $B_m B_n$. To be more precise, the first class of simple poles can be associated to the bound states B_{m+n} only for values of ξ such that $m + n < \mathrm{Int}(\pi/\xi)$. Indeed, (2.24) shows that outside this range the particle B_{m+n} is not in the spectrum of the model in spite of the fact that the pole in the amplitude (2.29) may still lie in the physical strip $\mathrm{Im}\,\theta \in (0, \pi)$. In this case, however, this pole can be explained in terms of a multiscattering process involving solitons as intermediate states. This is why, for generic values of ξ, the breather sector of the sine-Gordon model is not a self-contained bootstrap system.

[4]The rapidity variables θ_i parameterize energy and momentum of a particle as $(p_i^0, p_i^1) = (m\cosh\theta_i, m\sinh\theta_i)$, m being the mass. The scattering amplitudes depend on the rapidity difference between the colliding particles.

The situation becomes different when ξ takes the special values

$$\xi_N = \frac{2\pi}{2N+1}\,, \qquad N = 1, 2, \dots, \ . \tag{2.31}$$

In this case (2.24), (2.25) and (2.29) show that there are N breathers and that the formal identities

$$m_n = m_{2N+1-n} \tag{2.32}$$

$$S_{mn}(\theta) = S_{m,2N+1-n}(\theta) \tag{2.33}$$

hold, so that the pole discussed above can be associated to B_{m+n} for $m+n \leq N$ and to $B_{2N+1-m-n}$ for $m+n > N$, without any need to resort to the solitons. Hence, for the values (2.31) of the coupling, the breather sector of the sine-Gordon model provides alone a self-consistent factorized scattering theory and, consequently, defines an infinite series of massive integrable models labeled by the positive integer N. Equations (2.26) and (2.27) then identify these massive models with the minimal models $\mathcal{M}_{2/(2N+3)}$ with central charge

$$C_N = 1 - 3\,\frac{(2N+1)^2}{2N+3}\,, \tag{2.34}$$

perturbed by the operator $\phi_{1,3}$ with scaling dimension

$$X^{(N)}_{1,3} = -2\,\frac{2N-1}{2N+3}\ . \tag{2.35}$$

This conclusion was first reached in [11]. The thermodynamic Bethe ansatz [20, 30] confirms that the scattering theory (2.29) gives the central charges (2.34) for $\xi = \xi_N$, a result that can be regarded as a non-perturbative confirmation of the fact that the charge at infinity does not affect the dynamics of the particles. The minimal models $\mathcal{M}_{2/(2N+3)}$ possess the N non-trivial primary fields $\phi_{1,k}$, $k = 2, \dots, N+1$, plus the identity $\phi_{1,1}$. The negative values of the conformal data (2.34) and (2.35) show that these models do not satisfy reflection positivity. The case $N = 1$ corresponds to the Lee-Yang model [22 – 24], the simplest interacting quantum field theory. Its S-matrix $S_{11}(\theta)|_{\xi=2\pi/3} = t_{2/3}(\theta)$ was identified in [25].

3. Primary operators in the sinh-Gordon model

Most of the results discussed for the sine-Gordon model apply to the sinh-Gordon model (2.21) through the correspondence (2.23). In particular also the latter model is a massive integrable quantum field theory. The particle B interpolated by the scalar field corresponds to the sine-Gordon lightest breather B_1. The scattering amplitude

$$S(\theta) \equiv S_{11}(\theta) = t_{\frac{\xi}{\pi}}(\theta) \tag{3.1}$$

does not possess poles in the physical strip when b is real and completely specifies the S-matrix of the sinh-Gordon model. This amplitude was proposed and checked against

perturbation theory in b in [26 – 28]. It should be clear from the discussion of the previous section that the S-matrix is the same for the two ultraviolet limits (Gaussian fixed point and Liouville theory) compatible with the action (2.21).

The S-matrix determines the basic equations satisfied by the matrix elements of a local operator $\Phi(x)$ on the asymptotic multiparticle states [3, 4]. The form factors[5]

$$F_n^\Phi(\theta_1,\dots,\theta_n) = \langle 0|\Phi(0)|B(\theta_1)\dots B(\theta_n)\rangle \tag{3.2}$$

obey the equations

$$F_n^\Phi(\theta_1+\alpha,\dots,\theta_n+\alpha) = e^{s_\Phi \alpha} F_n^\Phi(\theta_1,\dots,\theta_n) \tag{3.3}$$

$$F_n^\Phi(\theta_1,\dots,\theta_i,\theta_{i+1},\dots,\theta_n) = S(\theta_i-\theta_{i+1})\, F_n^\Phi(\theta_1,\dots,\theta_{i+1},\theta_i,\dots,\theta_n) \tag{3.4}$$

$$F_n^\Phi(\theta_1+2i\pi,\theta_2,\dots,\theta_n) = F_n^\Phi(\theta_2,\dots,\theta_n,\theta_1) \tag{3.5}$$

$$\mathrm{Res}_{\theta'=\theta+i\pi}\, F_{n+2}^\Phi(\theta',\theta,\theta_1,\dots,\theta_n) = i\left[1-\prod_{j=1}^n S(\theta-\theta_j)\right] F_n^\Phi(\theta_1,\dots,\theta_n) \tag{3.6}$$

where the euclidean spin s_Φ is the only operator-dependent information.

The solutions of the equations (3.3)-(3.6) can be parameterized as [29, 21]

$$F_n^\Phi(\theta_1,\dots,\theta_n) = U_n^\Phi(\theta_1,\dots,\theta_n) \prod_{i<j} \frac{\mathcal{F}(\theta_i-\theta_j)}{\cosh\frac{\theta_i-\theta_j}{2}}\ , \tag{3.7}$$

Here the factors in the denominator introduce the annihilation poles prescribed by (3.6), and

$$\mathcal{F}(\theta) = \mathcal{N}(\xi)\, \exp\left[2\int_0^\infty \frac{dt}{t}\, q_\xi(t)\, \frac{\sinh\frac{t}{2}}{\sinh^2 t}\, \sin^2\frac{(i\pi-\theta)t}{2\pi}\right] \tag{3.8}$$

with

$$q_\xi(t) = -4\sinh\frac{\xi t}{2\pi}\, \sinh\left[\left(1+\frac{\xi}{\pi}\right)\frac{t}{2}\right] \tag{3.9}$$

$$\mathcal{N}(\xi) = \mathcal{F}(i\pi) = \exp\left[-\int_0^\infty \frac{dt}{t}\, q_\xi(t)\, \frac{\sinh\frac{t}{2}}{\sinh^2 t}\right]\ . \tag{3.10}$$

The function $\mathcal{F}(\theta)$ is the solution of the equations

$$\mathcal{F}(\theta) = S(\theta)\mathcal{F}(-\theta) \tag{3.11}$$

$$\mathcal{F}(\theta+2i\pi) = \mathcal{F}(-\theta) \tag{3.12}$$

with asymptotic behavior

$$\lim_{|\theta|\to\infty} \mathcal{F}(\theta) = 1\,; \tag{3.13}$$

it also satisfies the functional relation

$$\mathcal{F}(\theta+i\pi)\mathcal{F}(\theta) = \frac{\sinh\theta}{\sinh\theta-\sinh i\xi}\ . \tag{3.14}$$

The expression (3.8) is convergent in the range $-\pi < \xi < 0$ which is relevant for the sinh-Gordon model.

[5] We denote by $|0\rangle$ the vacuum state.

All the information about the operator is contained in the functions U_n^Φ. They must be entire functions of the rapidities, symmetric and (up to a factor $(-1)^{n-1}$) $2\pi i$-periodic in all θ_j. We write them in the form

$$U_n^\Phi(\theta_1,..,\theta_n) = \mathcal{H}_n \left(\frac{1}{\sigma_n^{(n)}}\right)^{(n-1)/2} Q_n^\Phi(\theta_1,..,\theta_n) \tag{3.15}$$

using the symmetric polynomials generated by

$$\prod_{i=1}^{n}(x+x_i) = \sum_{k=0}^{n} x^{n-k}\sigma_k^{(n)}(x_1,\ldots,x_n) \tag{3.16}$$

with $x_i \equiv e^{\theta_i}$, and choosing the constants

$$\mathcal{H}_n = \left(\frac{-8\sin\xi}{2^n \mathcal{F}(i\pi)}\right)^{n/2} . \tag{3.17}$$

The equations (3.3)–(3.6) imply

$$Q_n^\Phi(\theta_1+\alpha,..,\theta_n+\alpha) = e^{\left(s_\Phi+\frac{n(n-1)}{2}\right)\alpha} Q_n^\Phi(\theta_1,..,\theta_n) \tag{3.18}$$

$$Q_n^\Phi(\theta_1,..,\theta_i,\theta_{i+1},..,\theta_n) = Q_n^\Phi(\theta_1,..,\theta_{i+1},\theta_i,..,\theta_n) \tag{3.19}$$

$$Q_n^\Phi(\theta_1+2\pi i,..,\theta_n) = Q_n^\Phi(\theta_1,..,\theta_n) \tag{3.20}$$

$$Q_{n+2}^\Phi(\theta+i\pi,\theta,\theta_1,..,\theta_n) = (-1)^n x\, D_n(x,x_1,..,x_n)\, Q_n^\Phi(\theta_1,..,\theta_n) , \tag{3.21}$$

where $x \equiv e^\theta$ and

$$D_n(x,x_1,..,x_n) = \sum_{k=1}^{n}\sum_{m=1,odd}^{k} (-1)^{k+1}[m]\, x^{2(n-k)+m}\sigma_k^{(n)}\sigma_{k-m}^{(n)} , \tag{3.22}$$

with

$$[m] \equiv \frac{\sin(m\xi)}{\sin\xi} . \tag{3.23}$$

The equations (3.18)-(3.21) admit infinitely many solutions which account for the infinitely many operators with spin s_Φ. The scalar ($s_\Phi = 0$) solutions with the mildest asymptotic behavior are expected to correspond to the primary operators of the theory, namely the exponential operators (2.10). Introducing the notation

$$\hat{\Phi} = \frac{\Phi}{\langle\Phi\rangle} , \tag{3.24}$$

the asymptotic factorization condition

$$\lim_{\lambda\to+\infty} F_n^{\hat{\Phi}_0}(\theta_1+\lambda,\ldots,\theta_k+\lambda,\theta_{k+1},\ldots,\theta_n) = F_k^{\hat{\Phi}_0}(\theta_1,\ldots,\theta_k) F_{n-k}^{\hat{\Phi}_0}(\theta_{k+1},\ldots,\theta_n) \tag{3.25}$$

characterizes the scalar primary operators Φ_0 with non-vanishing matrix elements on any number of particles [31], and in this case selects the solutions [30]

$$Q_n^{(a)}(\theta_1,\ldots,\theta_n) = [a]\det M^{(n)}(a), \tag{3.26}$$

where $M^{(n)}(a)$ is the $(n-1)\times(n-1)$ matrix with entries

$$M^{(n)}_{i,j}(a) = [a+i-j]\,\sigma^{(n)}_{2i-j} \tag{3.27}$$

and a is a complex parameter. The trigonometric identity $[a]^2-[a-1][a+1]=1$ is useful to check that the functions (3.26) solve the recursive equation (3.21). It was found in [30] (see also [32]) that the solution (3.26) corresponds to the exponential operator[6] $\hat{V}_{iab}=\hat{V}_{-a\beta}$. The solutions (3.26) satisfy the property

$$Q^{(a)}_n = (-1)^n\, Q^{(-a)}_n\,, \tag{3.28}$$

which is expected since $a\to -a$ amounts to $\varphi\to-\varphi$, and the particle B is odd under this transformation. In presence of the charge at infinity $Q=ib\pi/\xi$ we expect the identification $\hat{V}_\alpha=\hat{V}_{Q-\alpha}$, namely

$$Q^{(a)}_n = Q^{(-a-\pi/\xi)}_n\ . \tag{3.29}$$

The identity

$$[a] = -[a+\pi/\xi]\,, \tag{3.30}$$

together with (3.28), ensures that (3.29) holds.

The solutions for the exponential operators allow to determine the form factors of the components of the energy-momentum tensor. Let us start with the trace (2.6). As seen in the previous section, when the ultraviolet limit is taken to be the Liouville theory (2.18) we have[7] $\Theta\sim V_{ib}$, so that

$$Q^{\Theta}_n = -\frac{\pi\, m^2}{8\sin\xi}\, Q^{(1)}_n\,, \tag{3.31}$$

where the normalization is fixed by the condition

$$F^{\Theta}_2(\theta+i\pi,\theta) = \frac{\pi}{2}\, m^2 \tag{3.32}$$

which corresponds to the normalization

$$\langle B(\theta)|B(\theta')\rangle = 2\pi\,\delta(\theta-\theta') \tag{3.33}$$

of the asymptotic states. If instead the sinh-Gordon model is seen as a perturbation of the Gaussian fixed point, the symmetric combination $Q^{(1)}_n+Q^{(-1)}_n$ must be considered. Then (3.28) shows that the result (3.31) still holds for n even, while Q^{Θ}_n vanishes for n odd. This analysis was first performed in [33], where the ultraviolet central charge was also evaluated through the C-theorem sum rule [17, 34]

$$C = \frac{12}{\pi}\int d^2x\,|x|^2\,\langle\Theta(x)\Theta(0)\rangle_{conn}\ . \tag{3.34}$$

It was checked that a truncated spectral expansion of the two-point trace correlator in terms of the form factors (3.31) reproduces with good approximation the Liouville central charge (2.9) if the sum is performed over all n, and the Gaussian value 1 if the sum is restricted to the even contributions [33].

[6] Since the identity $[a+2\pi/\xi]=[a]$ implies the periodicity $Q^{(a+2\pi/\xi)}_n = Q^{(a)}_n$, we consider values of a in the range $(0,2\pi/\xi)$.

[7] From now on we omit the tilde on the components of the energy-momentum tensor.

When inserted in the asymptotic factorization equation (3.25) with $\Phi = \Theta$ the solution (3.31) prescribes the result

$$\langle\Theta\rangle = F_0^\Theta = -\frac{\pi m^2}{8\sin\xi}\,, \tag{3.35}$$

which coincides with that known from the thermodynamic Bethe ansatz (see [35]).

The form factors of the other components of the energy-momentum tensor are easily obtained exploiting the conservation equations[8]

$$\begin{aligned}\bar\partial T &= \partial\Theta\\ \partial\bar T &= \bar\partial\Theta\end{aligned} \tag{3.36}$$

which lead to

$$\begin{aligned}F_n^T(\theta_1,\dots,\theta_n) &= -\frac{\sigma_1^{(n)}\sigma_n^{(n)}}{\sigma_{n-1}^{(n)}}\,F_n^\Theta(\theta_1,\dots,\theta_n)\\ F_n^{\bar T}(\theta_1,\dots,\theta_n) &= -\frac{\sigma_{n-1}^{(n)}}{\sigma_1^{(n)}\sigma_n^{(n)}}F_n^\Theta(\theta_1,\dots,\theta_n)\end{aligned} \tag{3.37}$$

for $n>0$; $\langle T\rangle = \langle\bar T\rangle = 0$ as for any operator with nonzero spin.

It follows from the discussion of the previous section that, through the analytic continuation (2.23), the above results for the exponential operators in the sinh-Gordon model also hold for the matrix elements of these operators on the breather B_1 of the sine-Gordon model. Moreover, when the coupling ξ takes the discrete values (2.31) corresponding to the reduction to the $\phi_{1,3}$-perturbed minimal models $\mathcal{M}_{2/(2N+3)}$, these results give, through the correspondence (2.17), the form factors for the independent primary operators $\phi_{1,l}$, $l=1,\dots,N+1$, of these massive minimal models. The reduction from the continuous spectrum of exponential operators of the sinh-Gordon model to the finite discrete spectrum of primary fields in the massive minimal models follows from the fact that in the latter case the form factors have to satisfy constraints on the bound state poles in addition to (3.3)-(3.6). For example, the fusion $B_1B_1\to B_2$ requires

$$\mathrm{Res}_{\theta'=\theta+i\xi}F_{n+2}^\Phi(\theta',\theta,\theta_1,\dots,\theta_n) = i\Gamma_{11}^2\langle 0|\Phi(0)|B_2(\theta'')B_1(\theta_1)\dots B_1(\theta_n)\rangle\,, \tag{3.38}$$

where $\theta''=\theta+i\xi/2$ and the three-particle coupling Γ_{11}^2 is obtained from

$$\mathrm{Res}_{\theta=i\xi}S_{11}(\theta) = i\left(\Gamma_{11}^2\right)^2\ . \tag{3.39}$$

It turns out [11, 36] that for $\xi=\xi_N$ the complete set of bound state equations implied by the breather sector seen as a self-contained bootstrap system selects among the solutions[9] (3.26) only those with $a=1,\dots,N$, besides the identity (see appendix E). These solutions

[8] We use the notation $\partial=\partial_z$ and $\bar\partial=\partial_{\bar z}$ with reference to the complex coordinates $z=x_1+ix_2$ and $\bar z = x_1-ix_2$.

[9] We stress that for $\xi=\xi_N$ the knowledge of the $Q_n^{(a)}$ for all n completely determines the operator since all matrix elements involving particles B_j with $j>1$ can be obtained through fusion equations like (3.38).

correspond to the operators

$$\hat{V}_{-k\beta_N} = \hat{\phi}_{1,2k+1}\,, \qquad k = 1, \dots, N \tag{3.40}$$

($\beta_N = -\sqrt{2/(2N+3)}$), which, in view of the reflection relation

$$\hat{\phi}_{1,l} = \hat{\phi}_{1,2N+3-l}\,, \qquad l = 1, \dots, 2N+2 \tag{3.41}$$

are all the primaries of this series of minimal models. The results (3.31), (3.35) and (3.37) for the matrix elements of the energy-momentum tensor apply to the minimal massive models for $\xi = \xi_N$.

4. The operator $T\bar{T}$

We have seen in the previous section how the primary operators correspond to the solutions of the form factor equations (3.3)-(3.6) with the mildest asymptotic behavior at high energies. The remaining solutions of these equations should span the space of descendant operators.

At criticality descendant operators are obtained acting on a primary with products of Virasoro generators L_{-i} and $\bar{L}_{-j}$. The sum of the positive integers i (j) defines the right (left) level l ($\bar{l}$) of the descendant. We denote by $\Phi_{l,\bar{l}}$ a descendant of level $(l,\bar{l})$ of a primary $\Phi_{0,0} \equiv \Phi_0$. Due to the isomorphism between critical and off-critical operator spaces, the notion of level holds also in the massive theory. A relation between the level and the asymptotic behavior of form factors has been introduced in [6, 7]. In particular, for operators $\Phi_{l,l}$ with non-zero matrix elements on any number of particles this relation reads

$$F_n^{\Phi_{l,l}}(\theta_1 + \lambda, \dots, \theta_k + \lambda, \theta_{k+1}, \dots, \theta_n) \sim e^{l\lambda}\,, \qquad \lambda \to +\infty \tag{4.1}$$

for $n > 1$ and $1 \le k \le n-1$. It was also argued that the scaling operators of this kind satisfy the asymptotic factorization property [6]

$$\begin{aligned} \lim_{\lambda\to+\infty} e^{-l\lambda} F_n^{\mathcal{L}_l\bar{\mathcal{L}}_l\Phi_0}(\theta_1 + \lambda, \dots, \theta_k + \lambda, \theta_{k+1}, \dots, \theta_n) = \\ = \frac{1}{\langle\Phi_0\rangle} F_k^{\mathcal{L}_l\Phi_0}(\theta_1, \dots, \theta_k)\, F_{n-k}^{\bar{\mathcal{L}}_l\Phi_0}(\theta_{k+1}, \dots, \theta_n)\,, \end{aligned} \tag{4.2}$$

where $\mathcal{L}_l$ and $\bar{\mathcal{L}}_l$ are operators that in the conformal limit converge to the product of right and left Virasoro generators, respectively, acting on the primary. This equation reduces to (3.25) in the case of primary operators.

This asymptotic information can be used to classify the form factor solutions according to the level and to determine the leading part in the high-energy (conformal) limit. The subleading contributions depend instead on the way the operators are defined off-criticality. A complete analysis of the operator space in the massive Lee-Yang model up to level 7 is given in [7].

The composite operator $T\bar{T}$, obtained from the non-scalar components of the energy-momentum tensor, is the simplest non-derivative scalar descendant of the identity. A. Zamolodchikov showed that this operator can be defined away from criticality as [8]

$$T\bar{T}(x) = \lim_{\epsilon\to 0}[T(x+\epsilon)\bar{T}(x) - \Theta(x+\epsilon)\Theta(x) + \text{derivative terms}]\,, \tag{4.3}$$

where 'derivative terms' means terms containing powers of ϵ times local operators which are total derivatives. One consequence of this equation is that, if $|n\rangle$ and $|m\rangle$ denote n- and m-particle states with the same energy ($E_n = E_m$) and momentum ($P_n = P_m$), the equation

$$\langle m|T\bar{T}(0)|n\rangle = \langle m|T(x)\bar{T}(0)|n\rangle - \langle m|\Theta(x)\Theta(0)|n\rangle \tag{4.4}$$

holds, with the r.h.s. that does not depend on x. Since the generic matrix element can be reduced to the form factors (3.2) iterating the crossing relation

$$\begin{aligned}\langle B(\theta'_m)\dots B(\theta'_1)|\Phi(0)|B(\theta_1)\dots B(\theta_n)\rangle = \langle B(\theta'_m)\dots B(\theta'_2)|\Phi(0)|B(\theta'_1+i\pi)B(\theta_1)\dots B(\theta_n)\rangle + \\ 2\pi\sum_{i=1}^{n}\delta(\theta'_1-\theta_i)\prod_{k=1}^{i-1}S(\theta_k-\theta'_1)\,\langle B(\theta'_m)\dots B(\theta'_2)|\Phi(0)|B(\theta_1)\dots B(\theta_{i-1})B(\theta_{i+1})\dots B(\theta_n)\rangle,\end{aligned} \tag{4.5}$$

the identities (4.4) contribute to the identification of the form factor solution for the operator $T\bar{T}$, in particular of the subleading parts which are left unconstrained by the asymptotic factorization property (4.2). Since $T\bar{T}$ is a level $(2,2)$ descendant of the identity, the factorization takes in this case the form

$$\lim_{\lambda\to+\infty} e^{-2\lambda}F_n^{T\bar{T}}(\theta_1+\lambda,\dots,\theta_k+\lambda,\theta_{k+1},\dots,\theta_n) = F_k^{T}(\theta_1,\dots,\theta_k)\,F_{n-k}^{\bar{T}}(\theta_{k+1},\dots,\theta_n)\,. \tag{4.6}$$

We now argue that (4.4) implies the property

$$F_n^{T\bar{T}} = \text{sum of terms containing } (\sigma_1^{(n)})^i(\sigma_{n-1}^{(n)})^j\,, \qquad i+j\geq 2 \tag{4.7}$$

for n larger than 2 but other than 4. Observe first that the components of the energy-momentum tensor, being local operators of the theory, must have form factors whose only singularities in rapidity space are the annihilation poles prescribed by (3.6), and possible bound state poles [37]. Then, it follows from (3.37) that

$$F_n^{T}\propto(\sigma_1^{(n)})^2\,, \qquad F_n^{\bar{T}}\propto(\sigma_{n-1}^{(n)})^2\,, \qquad F_n^{\Theta}\propto\sigma_1^{(n)}\sigma_{n-1}^{(n)}\,, \tag{4.8}$$

for $n>2$. Use now the resolution of the identity

$$I = \sum_{k=0}^{\infty}\frac{1}{k!}\int\frac{d\theta_1}{2\pi}\cdots\frac{d\theta_k}{2\pi}\,|k\rangle\langle k| \tag{4.9}$$

to expand the r.h.s. of (4.4) over matrix elements of T, $\bar{T}$ and Θ. If the total energy-momentum of the intermediate state $|k\rangle$ differs from that of $|m\rangle$ (which, we recall, equals that of $|n\rangle$) the two sums in the r.h.s. separately depend on x, and must cancel each

other in order to ensure the x-independence of the result. Then we are left with the contributions of matrix elements over states with the same energy and momentum, which all are x-independent. Consider the case m and n both larger than zero, $m \neq n$. It follows from (4.8) that each of these matrix elements generically vanishes at least as η^2, if η is an infinitesimal splitting between the energies of the two states in the matrix element. A milder behavior as $\eta \to 0$ is obtained when $|k\rangle$ is identical to $|m\rangle$ or to $|n\rangle$. Indeed it can be shown using (4.5) and (3.6) that the matrix elements of the energy-momentum tensor over identical states are finite and non-zero. In the r.h.s. of (4.4) these non-zero matrix elements multiply a matrix element vanishing at least as η^2. We argue in a moment that the form of the 'derivative terms' in (4.3) is such that they contribute terms vanishing at least as η^2 to the r.h.s. of (4.4). Putting all together, we conclude that the l.h.s. vanishes at least as η^2, and this implies the form (4.7). A more detailed derivation including the explanation of the limitations on n can be found in appendix B together with the form factor expansion of (4.4).

The property (4.7) can also be understood in the following way. Since $m\sigma_1^{(n)}$ and $m\sigma_{n-1}^{(n)}/\sigma_n^{(n)}$ are the eigenvalues of $P = i\partial$ and $\bar{P} = -i\bar{\partial}$ on an n-particle asymptotic state, (4.8) follows from the fact that in two dimensions the energy-momentum tensor can formally be written as $T_{\mu\nu}(x) = (2/\pi)(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)A(x)$, or

$$T = \partial^2 A\,, \qquad \bar{T} = \bar{\partial}^2 A\,, \qquad \Theta = \partial\bar{\partial} A\,, \tag{4.10}$$

in terms of an operator $A(x)$ which is *not* a local operator of the theory[10]. Using the notation $A \cdot B \equiv A(x+\epsilon)B(x)$ one has

$$T \cdot \bar{T} - \Theta \cdot \Theta = \frac{1}{2}\partial^2(A \cdot \bar{T}) + \frac{1}{2}\bar{\partial}^2(A \cdot T) - \partial\bar{\partial}(A \cdot \Theta)\ . \tag{4.11}$$

The property (4.7) then follows from the fact that also the 'derivative terms' in (4.3) must be derivative operators of at least second order: those associated to negative powers of ε because they must cancel the divergences arising in (4.11) when $\epsilon \to 0$; those associated to ϵ^0 because they must have $l = \bar{l} \leq 2$ and we know that $L_{-1} = \partial$, $\bar{L}_{-1} = \bar{\partial}$.

Having collected this information, we can move forward in the determination of the form factors of $T\bar{T}$. The first requirement to be satisfied is that they solve the form factor equations (3.3)-(3.6) and have the asymptotic behavior (4.1) with $l = 2$. Since the equations (3.3)-(3.6) are linear in the operator, the solution for $T\bar{T}$ can be written as a linear superposition of the scalar form factor solutions behaving as in (4.1) with $l = 0, 1, 2$. We will refer to these solutions as level 0, level 1 and level 2 solutions, respectively. We now argue that this linear combination can be restricted to

$$F_n^{T\bar{T}} = a\, m^{-2}\, F_n^{\partial^2\bar{\partial}^2\Theta} + F_n^{\mathcal{K}} + c\, F_n^{\partial\bar{\partial}\Theta} + d\, m^2\, F_n^{\Theta} + e\, m^4\, F_n^{I}\,, \tag{4.12}$$

with a, c, d, e dimensionless constants and $F_n^{\mathcal{K}}$ to be defined below. Indeed, the level 0 solutions corresponding to scaling operators are spanned by the primaries and, among

[10] Indeed, $F_2^A(\theta_1, \theta_2)$ contains a pole at $\theta_1 - \theta_2 = i\pi$, in contrast with (3.6) which prescribes a vanishing residue. Essentially, this is why (4.8) holds only for $n > 2$.

these, only the form factors

$$F_n^I = \delta_{n,0} \tag{4.13}$$

of the identity and those of Θ satisfy the requirement (4.7). As for the level 1 scaling operators, they are all of the form $L_{-1}\bar{L}_{-1}\Phi_0 = \partial\bar{\partial}\Phi_0$. Equations (4.4), (4.6) and (4.7) put no constraint on the contribution to (4.12) of such level 1 derivative operators. However, we expect that the form factors of the operator $T\bar{T}$ in the sinh-Gordon model enjoy the following properties[11]:

i) to give the form factors of $T\bar{T}$ on the lightest breather of the sine-Gordon model under analytic continuation to positive values of ξ;

ii) to give the form factors of $T\bar{T}$ on the lightest particle of the $\phi_{1,3}$-perturbed minimal models $\mathcal{M}_{2/(2N+3)}$ when we set ξ to the values ξ_N defined by (2.31);

iii) to be continuous functions of ξ.

We know that, while the sinh-Gordon and sine-Gordon models possess a continuous spectrum of primary operators, the $\phi_{1,3}$-perturbed minimal models $\mathcal{M}_{2/(2N+3)}$ possess a discrete spectrum of $N+1$ primaries. The identity and the trace of the energy-momentum tensor are the only primary operators which are present in all these models. Then the natural way to comply with the requirements i)-iii) is that I and Θ are the only primaries that contribute to (4.12). The extension of the argument to any of the models (2.28) with p/p' rational implies that this is actually the only possibility. While we had reached this conclusion about the primaries by another path, the present reasoning also requires that $\partial\bar{\partial}\Theta$ and $\partial^2\bar{\partial}^2\Theta$ are the only derivative operators which can appear in (4.12).

The superposition (4.12) without the term $F_n^{\mathcal{K}}$ is sufficient to provide the most general parameterization with the required asymptotic behavior up to $n = 2$. This is why $F_n^{\mathcal{K}}$ is a scalar three-particle kernel solution of the form factor equations, namely has the property

$$F_n^{\mathcal{K}} = 0\,, \qquad n = 0, 1, 2\,; \tag{4.14}$$

the first two non-vanishing elements of this solution are $F_3^{\mathcal{K}}$ and $F_4^{\mathcal{K}}$, with $U_3^{\mathcal{K}}$ and $U_4^{\mathcal{K}}$ which factorize $\prod_{i<j}\cosh\frac{\theta_i-\theta_j}{2}$ in such a way to satisfy (3.6) with 0 on the r.h.s. The $F_n^{\mathcal{K}}$ are made of terms which under the limit (4.1) behave as $e^{l\lambda}$ with $l = 0, 1, 2$. They define the local operator $\mathcal{K}(x)$ which in (4.12) accounts for the linear independence of $T\bar{T}$ within the operator space of the theory.

Since our equations do not constrain the level 1 derivative contributions to (4.12), the coefficient c will remain undetermined. This conclusion agrees with conformal pertubation theory, which states that, due to the resonance phenomenon [2] (see also [38]), the operator $T\bar{T}$ can only be defined up to a term proportional to $\partial\bar{\partial}\Theta(x)$ [8]. Our remaining task is that of showing that the r.h.s. of (4.12) can be uniquely determined up to this ambiguity.

For $n = 2$, the second derivative term is the only one contributing to the limit in (4.6), and this fixes

$$a = \frac{\langle\Theta\rangle}{m^2}\ . \tag{4.15}$$

[11] Analogous properties do hold for the components T, $\bar{T}$ and Θ of the energy-momentum tensor.

The coefficients d and e can be determined by (4.4) with $m = n = 1$. Indeed, when we use (4.9) to expand the operator product in the r.h.s., the x-independence of the result implies that only the $k = 1$ intermediate state gives a non-vanishing contribution. Using (4.5) to go to form factors we obtain the identities[12]

$$F_2^{T\bar{T}}(i\pi, 0) = -2\langle\Theta\rangle F_2^{\Theta}(i\pi, 0) = -\pi m^2 \langle\Theta\rangle \tag{4.16}$$

$$\langle T\bar{T}\rangle = -\langle\Theta\rangle^2 \ . \tag{4.17}$$

On the other hand, we have from (4.12) that

$$\langle T\bar{T}\rangle = d\, m^2 \, \langle\Theta\rangle + e\, m^4 \tag{4.18}$$

$$F_2^{T\bar{T}}(i\pi, 0) = d\, m^2 \, F_2^{\Theta}(i\pi, 0)\,, \tag{4.19}$$

so that we obtain

$$d = -\frac{2}{m^2}\,\langle\Theta\rangle \tag{4.20}$$

$$e = \frac{\langle\Theta\rangle^2}{m^4}\,. \tag{4.21}$$

The search for the kernel contribution $F_n^{\mathcal{K}}$ to (4.12) starts from the most general solution of the form factor equations satisfying (4.14), (4.7) and (4.1) with $l = 2$. We checked that equations (4.6) and (4.4) uniquely fix the level 2 and level 0 parts, respectively, within such a solution. Concerning the level 1 part, it cannot be determined by these conditions, because they do not exclude the contribution of those linear combinations of $\partial\bar{\partial}$-derivatives of the primaries which vanish on one- and two-particle states. This indetermination is eliminated if we impose the conditions i)-iii) above. To see this consider the $\phi_{1,3}$-perturbed minimal models $\mathcal{M}_{2/(2N+3)}$. The identification $B_{2N+1-n} \equiv B_n$, $n = 1, \ldots, N$, that follows from (2.33), implies the set of equations

$$\langle 0|\Phi(0)|B_{2N+1-n}\rangle = \langle 0|\Phi(0)|B_n\rangle \tag{4.22}$$

for any local operator Φ of the massive minimal model. In the minimal models, and more generally in the sine-Gordon model, the matrix elements involving particles B_n with $n > 1$ are related to the form factors (3.2) by residue equations on bound state poles of the type (3.38). Due to (2.33), in the minimal models these bound state equations constrain the form factors (3.2) themselves. For example, in the simplest case, $N = 1$, equation (3.38) holds with B_2 identified to B_1 (see appendix E for the case of generic N). We find that the only way of satisfying (4.22) for any N when $\Phi = T\bar{T}$ is to take[13]

$$\langle 0|\mathcal{K}(0)|B_n\rangle = 0 \tag{4.23}$$

[12]The identity (4.17) was originally observed in [38] and follows also from (4.4) with $m = n = 0$ and $|x| \to \infty$. See also [39] for results on the vacuum expectation values of descendant operators in integrable models.

[13]For $n = 1, 2$ (4.23) is implied by (4.14) for any value of ξ.

for $n = 1, \ldots, 2N$, or equivalently[14]

$$\lim_{\eta \to 0} \eta^{n-1} F_n^{\mathcal{K}} (\theta_1 + \eta, \theta_2 + 2\eta, \ldots, \theta_n + n\eta) = 0 \tag{4.24}$$

with $\theta_k = \theta_1 + i(k-1)\xi$, $k = 2, \ldots, n$ and $n \geq 1$.

The requirement of continuity in ξ then leads to extend (4.24) to generic values of ξ. We checked explicitly up to $n = 9$ that the conditions (4.4), (4.6) and (4.24) uniquely determine $F_n^{\mathcal{K}}$ for generic ξ. The explicit derivation up to $n = 4$ is given in appendix C, while appendix D contains the list of results up to $n = 7$. When inserted in (4.12) they provide the form factors of $T\bar{T}$ in the sinh-Gordon model seen as a perturbation of Liouville theory. The results apply to the sine-Gordon and the $\phi_{1,3}$-perturbed minimal models $\mathcal{M}_{2/(2N+3)}$ as specified by i) and ii) above. Setting to zero the $F_n^{T\bar{T}}$ with n odd one obtains the result for the sinh-Gordon and sine-Gordon models seen as perturbations of the Gaussian fixed point, for which the reflection symmetry $\varphi \to -\varphi$ holds. Few remarks about the free limit $\xi \to 0$ are contained in appendix F.

5. Conclusion

In this paper we identified the form factor solution corresponding to the operator $T\bar{T}$ in the sinh-Gordon model and in the $\phi_{1,3}$-perturbed minimal models $\mathcal{M}_{2/(2N+3)}$. The identification is obtained up to the arbitrary additive contribution of the operator $\partial\bar{\partial}\Theta$ which represents an intrinsic ambiguity in the definition of $T\bar{T}$.

We expect that the possibility of expressing the solution for $T\bar{T}$ as the superposition (4.12) of a three-particle kernel solution plus the contributions of the identity and the trace of the energy-momentum tensor together with its first two scalar derivatives is not specific to the class of models considered in this paper but is actually quite general. If so, the expressions of the coefficients a, d and e given here should be universal.

The property, expressed by (4.11) and (4.7), that $T\bar{T}$ behaves as a linear combination of derivative operators on states with more than four particles should also be very general. In particular, we remarked in [6] that it ensures that $T\bar{T}$ does not contribute to integrability breaking (at least to first order) when used to perturb a fixed point action. As a matter of fact, many examples are known of integrable massless flows in which $T\bar{T}$ is the leading operator driving the flow into the infrared fixed point [40].

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[14]The equivalence between (4.23) and (4.24) for $n > 2N$ follows from the periodicity at $\xi = \xi_N$.

A. Free boson

Consider the free massive boson described by the action (2.2) with

$$\mathcal{V} = -\frac{1}{2}m^2\varphi^2 \,. \tag{A.1}$$

The local scalar operators of the theory satisfy (3.3)–(3.6) with $s_\Phi = 0$ and $S(\theta) = 1$. The primary operators

$$V_\alpha = e^{i\sqrt{8\pi}\,\alpha\,\varphi} = \sum_{n=0}^{\infty} \frac{1}{n!}\,(i\sqrt{8\pi}\,\alpha\,\varphi)^n \tag{A.2}$$

are subject also to the factorization property (3.25). One then obtains

$$F_n^{V_\alpha} = \left(i\sqrt{8\pi}\,\alpha\,F_1^\varphi\right)^n \tag{A.3}$$

$$F_n^{\varphi^n} = n!\;(F_1^\varphi)^n \;\; . \tag{A.4}$$

Equation (2.7) gives in this free case[15]

$$\Theta = \Theta_{cl} = \frac{\pi m^2}{2}\left(\varphi^2 - \frac{iQ}{\sqrt{2\pi}}\,\varphi\right) , \tag{A.5}$$

and then

$$F_1^\Theta = -\frac{m^2}{2}\sqrt{\frac{\pi}{2}}\,iQ\,F_1^\varphi \tag{A.6}$$

$$F_2^\Theta = \pi m^2\;(F_1^\varphi)^2 \tag{A.7}$$

as the only non-zero form factors of this operator. Comparison with (3.32) fixes

$$F_1^\varphi = \frac{1}{\sqrt{2}} \,. \tag{A.8}$$

The results (2.9) and (2.11) then easily follow using (4.9) to evaluate over form factors the sum rules (3.34) and [31]

$$X_\Phi = -\frac{2}{\pi\langle\Phi\rangle}\int d^2x\,\langle\Theta(x)\Phi(0)\rangle_{conn} \,. \tag{A.9}$$

B. Constraints on the form factors of $T\bar{T}$

Here we derive the constraints imposed on the form factors of $T\bar{T}$ by the relations (4.4) and show how they lead to the property (4.7).

The r.h.s. of (4.4) is expanded introducing in between the two pairs of operators the resolution of the identity (4.9). Then the l.h.s and r.h.s. of (4.4) are rewritten in terms of form factors by iterative use of the crossing relation (4.5). Let $\langle m| = \langle B(\theta'_m)\dots B(\theta'_1)|$ and $|n\rangle = |B(\theta_1)\dots B(\theta_n)\rangle$ be the two states with the same energy and momentum in (4.4).

[15] We write Θ for $\tilde{\Theta}$.

In order to avoid to sit directly on annihilation or bound state poles of the form factors we introduce an infinitesimal splitting, parametrized by η, between the energies and momenta of these two states. The identity (4.4) is recovered in the limit $\eta \to 0$.

The l.h.s. has an expansion of the form:

$$\begin{aligned}
&\langle B(\theta'_m+\eta'_m)\dots B(\theta'_1+\eta'_1)|T\bar{T}(0)|B(\theta_1+\eta_1)\dots B(\theta_n+\eta_n)\rangle = \\
&F^{T\bar{T}}_{n+m}(\theta'_m+\eta'_m+i\pi,..,\theta'_1+\eta'_1+i\pi,\theta_1+\eta_1,..,\theta_n+\eta_n) + 2\pi\sum_{j=1}^{m}\sum_{i=1}^{n}\delta(\eta'_j-\eta_i+\theta'_j-\theta_i) \\
&\prod_{h=1}^{j-1}\prod_{k=1}^{i-1} S(\eta'_j-\eta'_h+\theta'_j-\theta'_h)\, S(\eta_k-\eta'_j+\theta_k-\theta'_j)\, F^{T\bar{T}}_{n+m-2}(\hat{\imath},\hat{\jmath}) + ... \qquad \text{(B.1)}
\end{aligned}$$

where $\eta'_a = (n+a)\eta$, $a = 1,\dots,m$, $\eta_b = b\eta$, $b = 1,\dots,n$, and $F^{T\bar{T}}_{n+m-2}(\hat{\imath},\hat{\jmath})$ is the form factor with $n+m-2$ particles obtained omitting the particles $B(\theta_i+\eta_i)$ and $B(\theta'_j+\eta'_j)$. The dots in (B.1) represent terms which factorize p delta functions, a product of two-particle amplitudes, and form factors of $T\bar{T}$ with $n+m-2p$ particles with $p = 2, ...,$ Int$((n+m)/2)$.

Let us consider now the r.h.s. of (4.4). Among the infinitely many terms generated by the insertion of (4.9) the only ones relevant for the identity (4.4) are those that do not depend on x for $\eta = 0$. For each fixed k, $\langle m|\, T(x)|k\rangle\, \langle k|\, \bar{T}(0)|n\rangle - \langle m|\, \Theta(x)|k\rangle\, \langle k|\, \Theta(0)|n\rangle$ can be expanded in terms of the form factors by using expansions like (B.1) for each matrix element. Here the arguments of the delta functions are the differences between the rapidities of the states $|m\rangle$ or $|n\rangle$ and those of $|k\rangle$. In these expansions the terms that do not depend on x are only those factorizing a set of delta functions which saturate all the integrations in (4.9) and fix the state $|k\rangle$ to one with the same energy and momentum of $|n\rangle$ and $|m\rangle$. Such terms are in finite number and are generated only if $k = n$ or $k = m$, when the delta functions fix the state $|k\rangle$ to $|n\rangle$ or $|m\rangle$, respectively. We can now rearrange such terms according to their content in delta functions: we have a number of terms which do not factorize delta functions plus a number of terms which factorize one delta function, and so on as in formula (B.1). The identity (4.4) is recovered imposing that in the l.h.s. and in the r.h.s. the terms that factorize the same delta functions coincide in the limit $\eta \to 0$. An iterative structure appears for these identities (see (4.16) for the particular case $m = n = 1$). In general, the terms factorizing delta functions give the results of the identities (4.4) with $m-1$ and $n-1$, while the terms without delta functions give the new condition

$$\begin{aligned}
&\lim_{\eta\to 0} F^{T\bar{T}}_{n+m}(\theta'_m+\eta'_m+i\pi,..,\theta'_1+\eta'_1+i\pi,\theta_1+\eta_1,..,\theta_n+\eta_n) = \\
&= \lim_{\eta\to 0}\{ \sum_{a=0}^{\min(n,m)} \frac{1}{2(n-a)!a!} \sum_{\tau\in\mathcal{P}_n}\sum_{\mu\in\mathcal{P}_m}\prod_{h=1}^{a}\Delta(\theta'_{\mu_{m-a+h}}-\theta_{\tau_{n-a+h}}) \\
&\times\{[F^{T}_{m+n-2a}(\theta'_{\mu_{m-a}}+\eta'_{\mu_{m-a}}+i\pi,..,\theta'_{\mu_1}+\eta'_{\mu_1}+i\pi,\theta_{\tau_1}+\eta_{\tau_1},..,\theta_{\tau_{n-a}}+\eta_{\tau_{n-a}})\times \\
&F^{\bar{T}}_{2a}(\theta'_{\mu_m}+\eta'_{\mu_m}+i\pi,..,\theta'_{\mu_{m-a+1}}+\eta'_{\mu_{m-a+1}}+i\pi,\theta_{\tau_{n-a+1}}+\eta_{\tau_{n-a+1}},..,\theta_{\tau_n}+\eta_{\tau_n}) - \\
&F^{\Theta}_{m+n-2a}(\theta'_{\mu_{m-a}}+\eta'_{\mu_{m-a}}+i\pi,..,\theta'_{\mu_1}+\eta'_{\mu_1}+i\pi,\theta_{\tau_1}+\eta_{\tau_1},..,\theta_{\tau_{n-a}}+\eta_{\tau_{n-a}})\times \\
&F^{\Theta}_{2a}(\theta'_{\mu_m}+\eta'_{\mu_m}+i\pi,..,\theta'_{\mu_{m-a+1}}+\eta'_{\mu_{m-a+1}}+i\pi,\theta_{\tau_{n-a+1}}+\eta_{\tau_{n-a+1}},..,\theta_{\tau_n}+\eta_{\tau_n})]+[T\leftrightarrow\bar{T}]\} \qquad \text{(B.2)}
\end{aligned}$$

where $\mathcal{P}_n$ is the group of permutations of n indices, τ is a permutation and

$$\Delta(x) = \begin{cases} 1 & \text{for } x = 0 \ , \\ 0 & \text{for } x \neq 0 \ . \end{cases}$$

Let us observe now that the r.h.s. of (B.2) goes always to zero at least as η^2, because the p-particle form factors of $T, \bar{T}$ and Θ respectively factorize $(\sigma_1^{(p)})^2$, $(\sigma_{p-1}^{(p)})^2$ and $\sigma_1^{(p)}\sigma_{p-1}^{(p)}$. The only exceptions to this situation arise if some rapidities are grouped in pairs that lie on annihilation poles or for rapidity configurations which intercept bound state poles. In every case, in terms of the parametrization (3.7), the only way in which the form factors of $T\bar{T}$ can satisfy (B.2) is that for any $p > 2$ the function $U_p^{T\bar{T}}$ is the sum of terms each one factorizing $\left(\sigma_1^{(p)}\right)^i \left(\sigma_{p-1}^{(p)}\right)^j$ with some i and j such that $i + j \geq 2$.

This property amounts to (4.7) provided that there are no cancellations with the denominator of (3.7). This denominator can be written as

$$\begin{aligned} \prod_{1\leq i<j\leq p} \cosh(\theta_i - \theta_j) &= \left(\frac{1}{2^p \sigma_p^{(p)}}\right)^{(p-1)/2} \prod_{1\leq i<j\leq p} (x_i + x_j) \\ &= \left(\frac{1}{2^p \sigma_p^{(p)}}\right)^{(p-1)/2} \det D^{(p)} \, , \end{aligned} \tag{B.3}$$

where $D^{(p)}$ is the $(p-1) \times (p-1)$ matrix with entries

$$D_{ij}^{(p)} = \sigma_{2i-j}^{(p)} \ . \tag{B.4}$$

For $p > 2$, it is only for $p = 4$ that (B.3) is a sum of terms all containing non-vanishing powers of $\sigma_1^{(p)}$ or $\sigma_{p-1}^{(p)}$ and cancellations may occur.

C. Explicit construction

In this appendix we determine the form factors of the kernel $\mathcal{K}$ up to four particles. At three particles the most general solution of the form factor equations satisfying (4.7), (4.1) with $l = 2$ and the initial condition

$$F_1^{\mathcal{K}} = 0 \tag{C.1}$$

is[16]

$$\frac{b_3\sigma_1^2\sigma_2^2 + \mathrm{D}_3\sigma_2^3 + \mathrm{C}_3\sigma_1^3\sigma_3}{\sigma_3^2} F_3^{K_3} + \mathrm{B}_3 \frac{\sigma_1\sigma_2}{\sigma_3} F_3^{K_3} \ . \tag{C.2}$$

Here, $F_3^{K_3}$ is defined by

$$Q_3^{K_3} = (\sigma_1\sigma_2 - \sigma_3) \tag{C.3}$$

in the parametrization (3.7) and generates the one dimensional space of the level 0 solutions of the equations (3.3)-(3.6) which vanish at one particle. The first and second term in (C.2) are pure level 2 and level 1 parts, respectively, and b_3, B_3, C_3 and D_3 are dimensionless

[16] We simplify the notation by dropping the superscript (n) on the symmetric polynomials.

coefficients. The asymptotic factorization property fixes the level 2 solution, i.e. the coefficients b_3, C_3 and D_3 to

$$b_3 = b = -\langle\Theta\rangle^2\,, \tag{C.4}$$

$$C_3 = -b\,,\ D_3 = -b\,. \tag{C.5}$$

For $n = 3$ the condition (4.24) fixes the level 1 part, i.e. the coefficient B_3 to

$$B_3 = -b(1 + 2\cos 2\xi). \tag{C.6}$$

Now, let us consider the four-particle case. The most general level 2 solution to the form factor equations satisfying (4.7) and the initial conditions

$$F_2^{\mathcal{K}} = 0 \tag{C.7}$$

is

$$\frac{b_4\sigma_1^2\sigma_3^2 + E_4\sigma_1^2\sigma_2\sigma_4 + D_4\sigma_2\sigma_3^2 + C_4\sigma_2^2\sigma_4}{\sigma_4^2}F_4^{K_4} + B_4\,\frac{\sigma_1\sigma_3}{\sigma_4}F_4^{K_4} + A_4F_4^{K_4}\ . \tag{C.8}$$

Here, $F_4^{K_4}$ is defined by

$$Q_4^{K_4} = (\sigma_1\sigma_3\sigma_2 - \sigma_3^2 - \sigma_1^2\sigma_4)$$

and generates the one dimensional space of the level 0 solutions of the equations (3.3)-(3.6) which vanish at two particles. The asymptotic factorization property fixes the level 2 part of (C.8), i.e. the coefficients b_4 to the same value of b, and C_4, D_4 and E_4 to

$$C_4 = -b\,,\ D_4 = 0\,,\ E_4 = 0\ . \tag{C.9}$$

The general result (B.2) for $n = m = 2$ is rewritten as

$$\begin{aligned}\lim_{\eta\to 0} F_4^{T\bar{T}}(\theta_2+\eta+i\pi,\theta_1+\eta+i\pi,\theta_1,\theta_2) = \lim_{\eta\to 0}\{&[F_2^T(\theta_1+\eta+i\pi,\theta_1)F_2^{\bar{T}}(\theta_2+\eta+i\pi,\theta_2)\\ &-F_2^{\Theta}(\theta_1+\eta+i\pi,\theta_1)F_2^{\Theta}(\theta_2+\eta+i\pi,\theta_2) - \langle\Theta\rangle F_4^{\Theta}(\theta_2+\eta+i\pi,\theta_1+\eta+i\pi,\theta_1,\theta_2)]\\ &+[T\leftrightarrow\bar{T}]\}\,.\end{aligned} \tag{C.10}$$

This expression implies for the four-particle kernel

$$\lim_{\eta\to 0} F_4^{\mathcal{K}}(\theta_2+\eta+i\pi,\theta_1+\eta+i\pi,\theta_1,\theta_2) = 2[F_2^{\Theta}(i\pi,0)]^2\,(\cosh 2(\theta_1-\theta_2)-1)\,, \tag{C.11}$$

where the r.h.s. comes from the identity

$$F_2^T(\theta_1+i\pi,\theta_1)F_2^{\bar{T}}(\theta_2+i\pi,\theta_2) - F_2^{\Theta}(\theta_1+i\pi,\theta_1)F_2^{\Theta}(\theta_2+i\pi,\theta_2) = [e^{2(\theta_1-\theta_2)}-1]\,[F_2^{\Theta}(i\pi,0)]^2. \tag{C.12}$$

The above condition fixes the level 0 part of (C.8), i.e. the coefficient A_4 to

$$A_4 = 4b\cos^2\xi\ . \tag{C.13}$$

Finally, for $n = 4$ the condition (4.24) fixes the level 1 part, i.e. the coefficient B_4 to

$$B_4 = -2b\cos\xi(1 + 4\cos\xi + 2\cos 2\xi). \tag{C.14}$$

This procedure has been implemented up to nine particles and the result is given in the next appendix.

D. Kernel solution

We list in this appendix the functions $\tilde{Q}_n^{\mathcal{K}}$ which through (3.7), (3.15) and

$$Q_n^{\mathcal{K}} = -\langle\Theta\rangle^2\, \tilde{Q}_n^{\mathcal{K}} \tag{D.1}$$

determine $F_n^{\mathcal{K}}$ in (4.12). The $\tilde{Q}_n^{\mathcal{K}}$ have been determined explicitly up to $n = 9$. The functions $\tilde{Q}_8^{\mathcal{K}}$ and $\tilde{Q}_9^{\mathcal{K}}$, however, are too cumbersome and we do not reproduce them here. We simplify the notation by dropping the superscript (n) on the symmetric polynomials.

$$\tilde{Q}_3^{\mathcal{K}} = \frac{1}{\sigma_3^2}(\sigma_1^2\sigma_2^2 - \sigma_2^3 - \sigma_1^3\sigma_3 - (1+2\cos[2\xi])\,\sigma_1\sigma_2\sigma_3)(\sigma_1\sigma_2 - \sigma_3) \tag{D.2}$$

$$\begin{aligned}\tilde{Q}_4^{\mathcal{K}} = \frac{1}{\sigma_4^2}&(\sigma_1^2\sigma_3^2 - \sigma_2^2\sigma_4 - 2\cos[\xi](1+4\cos[\xi]+2\cos[2\xi])\,\sigma_1\sigma_3\sigma_4 + 4\cos[\xi]^2\sigma_4^2)\\ &(\sigma_1\sigma_3\sigma_2 - \sigma_3^2 - \sigma_1^2\sigma_4)\end{aligned} \tag{D.3}$$

$$\begin{aligned}\tilde{Q}_5^{\mathcal{K}} = \frac{1}{\sigma_5^2}&(-\sigma_1^4\sigma_3\sigma_4^2\sigma_5 + \sigma_4^2\sigma_5(\sigma_2^2\sigma_3 - 4\cos[\xi]^2\sigma_3\sigma_4 - \sigma_2\sigma_5) + \sigma_1\sigma_4\sigma_5(-\sigma_2^3\sigma_4 + \sigma_2(2(2+\cos[2\xi])\sigma_4^2\\ &-(7+8\cos[\xi]+8\cos[2\xi]+4\cos[3\xi]+2\cos[4\xi])\sigma_3\sigma_5) + 4\cos[\xi]((1+4\cos[\xi]+2\cos[2\xi]\\ &+\cos[3\xi])\sigma_3^2\sigma_4 + (1+2\cos[\xi]+2\cos[2\xi])\sigma_5^2)) + \sigma_1^2(2(4+4\cos[\xi]+5\cos[2\xi]+2\cos[3\xi]\\ &+\cos[4\xi])\sigma_2^2\sigma_4\sigma_5^2 - \sigma_5(\sigma_3^3\sigma_4 + 8\cos[\frac{\xi}{2}]^2(2+\cos[\xi]+2\cos[2\xi]+\cos[3\xi])\sigma_4^2\sigma_5 + \sigma_3\sigma_5^2)\\ &-\sigma_2(\sigma_4^4 + 2(5+4\cos[\xi]+6\cos[2\xi]+2\cos[3\xi]+\cos[4\xi])\sigma_3\sigma_4^2\sigma_5 - \sigma_3^2\sigma_5^2))\\ &+\sigma_1^3(2\sigma_4\sigma_5((3+4\cos[\xi]+4\cos[2\xi]+2\cos[3\xi]+\cos[4\xi])\sigma_4^2 + (2+\cos[2\xi])\sigma_3\sigma_5)\\ &+\sigma_2(\sigma_3\sigma_4^3 - 4\cos[\xi]^2\sigma_5^3)))\end{aligned} \tag{D.4}$$

$$\begin{aligned}\tilde{Q}_6^{\mathcal{K}} = \frac{1}{\sigma_6^2}&(\sigma_1^3\sigma_2\sigma_3\sigma_4\sigma_5^3 - \sigma_1^2\sigma_2\sigma_3\sigma_5^4 - 4\cos[\xi]^2\sigma_1^3\sigma_4\sigma_5^4 + 4\cos[\xi]^2\sigma_1^2\sigma_5^5 - \sigma_1^2\sigma_2\sigma_4^3\sigma_5\sigma_6 - \sigma_1^2\sigma_3^3\sigma_5^2\sigma_6\\ &-\sigma_1\sigma_2^3\sigma_4\sigma_5^2\sigma_6 - \sigma_1^4\sigma_3\sigma_4\sigma_5^2\sigma_6 - 2\cos[\xi](7+16\cos[\xi]+12\cos[2\xi]+4\cos[3\xi]\\ &+2\cos[4\xi])\sigma_1^2\sigma_2\sigma_3\sigma_4\sigma_5^2\sigma_6 + \sigma_2^2\sigma_3\sigma_4\sigma_5^2\sigma_6 + 2(6+11\cos[\xi]+8\cos[2\xi]+6\cos[3\xi]+2\cos[4\xi]\\ &+\cos[5\xi])\sigma_1\sigma_3^2\sigma_4\sigma_5^2\sigma_6 + 2(6+11\cos[\xi]+8\cos[2\xi]+6\cos[3\xi]+2\cos[4\xi]+\cos[5\xi])\sigma_1^3\sigma_4^2\sigma_5^2\sigma_6\\ &+2(2+\cos[2\xi])\sigma_1\sigma_2\sigma_4^2\sigma_5^2\sigma_6 - 4\cos[\xi]^2\sigma_3\sigma_4^2\sigma_5^2\sigma_6 - 4\cos[\xi]^2\sigma_1^4\sigma_2\sigma_5^3\sigma_6 + 2(6+11\cos[\xi]+8\cos[2\xi]\\ &+6\cos[3\xi]+2\cos[4\xi]+\cos[5\xi])\sigma_1^2\sigma_2^2\sigma_5^3\sigma_6 + (5+4\cos[2\xi])\sigma_1^3\sigma_3\sigma_5^3\sigma_6 - 2(1+2\cos[\xi])^3(1-\cos[\xi]\\ &+\cos[2\xi])\sigma_1\sigma_2\sigma_3\sigma_5^3\sigma_6 - 2(6+15\cos[\xi]+8\cos[2\xi]+8\cos[3\xi]+2\cos[4\xi]+\cos[5\xi])\sigma_1^2\sigma_4\sigma_5^3\sigma_6\\ &-\sigma_2\sigma_4\sigma_5^3\sigma_6 + 4(2\cos[\xi]+\cos[3\xi])\sigma_1\sigma_5^4\sigma_6 + \sigma_1^2\sigma_2\sigma_3\sigma_4^2\sigma_6^2 + 2(6+11\cos[\xi]+8\cos[2\xi]+6\cos[3\xi]\\ &+2\cos[4\xi]+\cos[5\xi])\sigma_1^2\sigma_2\sigma_3^2\sigma_5\sigma_6^2 - 2(1+2\cos[\xi])^3(1-\cos[\xi]+\cos[2\xi])\sigma_1\sigma_3^3\sigma_5\sigma_6^2 + 2(2\\ &+\cos[2\xi])\sigma_1^2\sigma_2^2\sigma_4\sigma_5\sigma_6^2 - 2(1+2\cos[\xi])^3(1-\cos[\xi]+\cos[2\xi])\sigma_1^3\sigma_3\sigma_4\sigma_5\sigma_6^2 - 2\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6^2\\ &-(3+4\cos[2\xi])\sigma_1^2\sigma_4^2\sigma_5\sigma_6^2 + 4\cos[\xi]^2\sigma_1^5\sigma_5^2\sigma_6^2 - 2(6+15\cos[\xi]+8\cos[2\xi]+8\cos[3\xi]+2\cos[4\xi]\\ &+\cos[5\xi])\sigma_1^3\sigma_2\sigma_5^2\sigma_6^2 - (3+4\cos[2\xi])\sigma_1\sigma_2^2\sigma_5^2\sigma_6^2 + 2(4+17\cos[\xi]+6\cos[2\xi]+9\cos[3\xi]+2\cos[4\xi]\\ &+\cos[5\xi])\sigma_1^2\sigma_3\sigma_5^2\sigma_6^2 + (1+2\cos[2\xi])\sigma_2\sigma_3\sigma_5^2\sigma_6^2 - 2(1+\cos[2\xi]+\cos[4\xi])\sigma_1\sigma_4\sigma_5^2\sigma_6^2 + (1\\ &+2\cos[2\xi])^2\sigma_5^3\sigma_6^2 - 4\cos[\xi]^2\sigma_1^2\sigma_2^2\sigma_3\sigma_6^3 - \sigma_1^3\sigma_2\sigma_4\sigma_6^3 + (1+2\cos[2\xi])\sigma_1^2\sigma_3\sigma_4\sigma_6^3 + 4(2\cos[\xi]\\ &+\cos[3\xi])\sigma_1^4\sigma_5\sigma_6^3 - 2(1+\cos[2\xi]+\cos[4\xi])\sigma_1^2\sigma_2\sigma_5\sigma_6^3 + 2(1+2\cos[2\xi])^2\sigma_1\sigma_3\sigma_5\sigma_6^3\end{aligned}$$

$$+(1+2\cos[2\xi])^2\sigma_1^3\sigma_6^4) \tag{D.5}$$

$$\tilde{Q}_7^{\mathcal{K}} = \frac{1}{\sigma_7^2}(\sigma_1^5\sigma_6^2\sigma_7(4\cos[\xi]^2\sigma_5\sigma_6 - (1+2\cos[2\xi])\sigma_4\sigma_7) - \sigma_1^4\sigma_6(-\sigma_7(-\sigma_3\sigma_4\sigma_5\sigma_6 + (3+8\cos[\xi]$$
$$+4\cos[2\xi]+4\cos[3\xi]+2\cos[4\xi])\sigma_6^3 - 2(6+7\cos[2\xi]+2\cos[4\xi])\sigma_5\sigma_6\sigma_7$$
$$+8\cos[\xi]^2(1+2\cos[2\xi])\sigma_4\sigma_7^2) + \sigma_2\sigma_6(4\cos[\xi]^2\sigma_5\sigma_6^2 - \sigma_7((1+2\cos[2\xi])\sigma_4\sigma_6 + 4\cos[\xi]^2\sigma_3\sigma_7)))$$
$$-\sigma_6^2\sigma_7(4\cos[\xi]^2\sigma_2^3\sigma_3\sigma_7 - \sigma_2^2\sigma_5(\sigma_3\sigma_4 + 4\cos[\xi]^2\sigma_7) - \sigma_5((1+2\cos[2\xi])^2\sigma_5\sigma_6 - 8\cos[\xi]^2\cos[2\xi]\sigma_4\sigma_7)$$
$$+4\cos[\xi]^2\sigma_3(\sigma_4^2\sigma_5 + 4\cos[\xi]^2(1+2\cos[2\xi])\sigma_6\sigma_7) + \sigma_2(-(1+2\cos[2\xi])\sigma_3\sigma_5\sigma_6 + 4\cos[\xi]^2\sigma_7^2$$
$$+\sigma_4(\sigma_5^2 - 16\cos[\xi]^4\sigma_3\sigma_7))) + \sigma_1\sigma_6\sigma_7(-\sigma_2^3\sigma_4\sigma_5\sigma_6 - 4\cos[\xi]^2(-1+2\cos[\xi])(1+2\cos[\xi])^3\sigma_3^3\sigma_6^2$$
$$+4\cos[\xi]^2\sigma_2^4\sigma_6\sigma_7 - 4\cos[\xi]\sigma_3^2\sigma_4(-(5+13\cos[\xi]+8\cos[2\xi]+6\cos[3\xi]+2\cos[4\xi]+\cos[5\xi])\sigma_5\sigma_6$$
$$+(1+2\cos[\xi])^3(1-\cos[\xi]+\cos[2\xi])\sigma_4\sigma_7) + (1+2\cos[2\xi])\sigma_3(8\cos[\xi]^2\sigma_6^3 + (11+20\cos[2\xi]$$
$$+8\cos[3\xi]+10\cos[4\xi]+4\cos[5\xi]+2\cos[6\xi])\sigma_5\sigma_6\sigma_7 - 8\cos[\xi]^2(1+3\cos[2\xi]+2\cos[3\xi]$$
$$+\cos[4\xi])\sigma_4\sigma_7^2) - \sigma_2^2((3+4\cos[2\xi])\sigma_5\sigma_6^2 + \sigma_7((9+10\cos[2\xi]+2\cos[4\xi])\sigma_4\sigma_6 - 8\cos[\xi]^2\sigma_3\sigma_7))$$
$$+\sigma_2(2(2+\cos[2\xi])\sigma_4^2\sigma_5\sigma_6 - 4\cos[\xi]^2(-1+2\cos[\xi])(1+2\cos[\xi])^3\sigma_3\sigma_5^2\sigma_6 + (11+20\cos[\xi]$$
$$+20\cos[2\xi]+12\cos[3\xi]+10\cos[4\xi]+4\cos[5\xi]+2\cos[6\xi])\sigma_3\sigma_4\sigma_5\sigma_7 + 8\cos[\xi]^2(1+4\cos[\xi]$$
$$+3\cos[2\xi]+2\cos[3\xi]+\cos[4\xi])\sigma_3^2\sigma_6\sigma_7 - \sigma_7(-(19+26\cos[2\xi]+10\cos[4\xi]+2\cos[6\xi])\sigma_6^2$$
$$+4(3+5\cos[\xi]+4\cos[2\xi]+3\cos[3\xi]+\cos[4\xi]+\cos[5\xi])\sigma_5\sigma_7)) + 2(2(2\cos[\xi]+\cos[3\xi])\sigma_5^3\sigma_6$$
$$-(3+3\cos[2\xi]+\cos[4\xi])\sigma_4\sigma_5\sigma_6^2 + 4\cos[\xi]^2\sigma_4\sigma_5^2\sigma_7 + 2\cos[\xi]^2(1+2\cos[2\xi])\sigma_7(\sigma_4^2\sigma_6 + 2(1$$
$$+2\cos[\xi]+2\cos[2\xi]+2\cos[3\xi])\sigma_7^2))) + \sigma_1^3(2(13+18\cos[\xi]+19\cos[2\xi]+10\cos[3\xi]+7\cos[4\xi]$$
$$+2\cos[5\xi]+\cos[6\xi])\sigma_4^2\sigma_5\sigma_6^2\sigma_7 - 4\cos[\xi]^2(-1+2\cos[\xi])(1+2\cos[\xi])^3\sigma_4^3\sigma_6\sigma_7^2$$
$$+\sigma_2^2(-4\cos[\xi]^2\sigma_3\sigma_6^3\sigma_7 + (1+2\cos[2\xi])^2\sigma_7^4) + \sigma_6(-(1+2\cos[2\xi])\sigma_3^2\sigma_6\sigma_7^2 + \sigma_3\sigma_5\sigma_6((1$$
$$+2\cos[2\xi])\sigma_6^2 + 2(2+\cos[2\xi])\sigma_5\sigma_7) + \sigma_7^2(2(13+14\cos[\xi]+22\cos[2\xi]+12\cos[3\xi]$$
$$+13\cos[4\xi]+8\cos[5\xi]+5\cos[6\xi]+2\cos[7\xi]+\cos[8\xi])\sigma_6^2 + (19+26\cos[2\xi]+10\cos[4\xi]$$
$$+2\cos[6\xi])\sigma_5\sigma_7)) - 4\cos[\xi]^2\sigma_4(\sigma_5^2\sigma_6^3 + (1+2\cos[2\xi])\sigma_7((1+2\cos[\xi])^2\sigma_3\sigma_6^3 + 4\cos[\xi]^2\sigma_7^3))$$
$$+\sigma_2(4\cos[\xi]^2\sigma_6^5 - 2(7+22\cos[\xi]+11\cos[2\xi]+12\cos[3\xi]+5\cos[4\xi]+2\cos[5\xi]$$
$$+\cos[6\xi])\sigma_5\sigma_6^3\sigma_7 - (3+4\cos[2\xi])\sigma_5^2\sigma_6\sigma_7^2 + 2(7+22\cos[\xi]+11\cos[2\xi]+12\cos[3\xi]+5\cos[4\xi]$$
$$+2\cos[5\xi]+\cos[6\xi])\sigma_4\sigma_6^2\sigma_7^2 + (1+2\cos[2\xi])\sigma_4\sigma_5\sigma_7^3 + \sigma_3\sigma_6(\sigma_4\sigma_5\sigma_6^2 - 2(3+3\cos[2\xi]$$
$$+\cos[4\xi])\sigma_7^3))) + \sigma_1^2(-\sigma_3\sigma_6^5 - 2\cos[2\xi]\sigma_3\sigma_6^5 + 2\sigma_5^4\sigma_6\sigma_7 + 2\cos[2\xi]\sigma_5^4\sigma_6\sigma_7 - \sigma_3^3\sigma_5\sigma_6^2\sigma_7$$
$$+14\sigma_3\sigma_5\sigma_6^3\sigma_7 + 44\cos[\xi]\sigma_3\sigma_5\sigma_6^3\sigma_7 + 22\cos[2\xi]\sigma_3\sigma_5\sigma_6^3\sigma_7 + 24\cos[3\xi]\sigma_3\sigma_5\sigma_6^3\sigma_7$$
$$+10\cos[4\xi]\sigma_3\sigma_5\sigma_6^3\sigma_7 + 4\cos[5\xi]\sigma_3\sigma_5\sigma_6^3\sigma_7 + 2\cos[6\xi]\sigma_3\sigma_5\sigma_6^3\sigma_7 - 9\sigma_3\sigma_5^2\sigma_6\sigma_7^2$$
$$-10\cos[2\xi]\sigma_3\sigma_5^2\sigma_6\sigma_7^2 - 2\cos[4\xi]\sigma_3\sigma_5^2\sigma_6\sigma_7^2 + 4(2\cos[\xi]+\cos[3\xi])\sigma_2^3\sigma_6\sigma_7^3 + 4\sigma_3^2\sigma_6\sigma_7^3$$
$$+6\cos[2\xi]\sigma_3^2\sigma_6\sigma_7^3 + 2\cos[4\xi]\sigma_3^2\sigma_6\sigma_7^3 - 49\sigma_6^2\sigma_7^3 - 80\cos[\xi]\sigma_6^2\sigma_7^3 - 80\cos[2\xi]\sigma_6^2\sigma_7^3$$
$$-60\cos[3\xi]\sigma_6^2\sigma_7^3 - 44\cos[4\xi]\sigma_6^2\sigma_7^3 - 32\cos[5\xi]\sigma_6^2\sigma_7^3 - 14\cos[6\xi]\sigma_6^2\sigma_7^3 - 8\cos[7\xi]\sigma_6^2\sigma_7^3$$
$$-2\cos[8\xi]\sigma_6^2\sigma_7^3 - 2\sigma_5\sigma_7^4 - 2\cos[2\xi]\sigma_5\sigma_7^4 - \sigma_4^2\sigma_6\sigma_7((1+2\cos[2\xi])\sigma_6^2 - 8\cos[\xi]^2(1+4\cos[\xi]$$
$$+3\cos[2\xi]+2\cos[3\xi]+\cos[4\xi])\sigma_5\sigma_7) + \sigma_2^2\sigma_7(2(13+18\cos[\xi]+19\cos[2\xi]+10\cos[3\xi]+7\cos[4\xi]$$
$$+2\cos[5\xi]+\cos[6\xi])\sigma_5^2\sigma_6^2 - \sigma_5\sigma_7(4\cos[\xi]^2(-1+2\cos[\xi])(1+2\cos[\xi])^3\sigma_4\sigma_6 + \sigma_3\sigma_7) + 2\sigma_6^2((2$$
$$+\cos[2\xi])\sigma_4\sigma_6 - (10+22\cos[\xi]+15\cos[2\xi]+12\cos[3\xi]+6\cos[4\xi]+2\cos[5\xi]+\cos[6\xi])\sigma_3\sigma_7))$$
$$-4\cos[\xi]\sigma_4((6+10\cos[\xi]+10\cos[2\xi]+5\cos[3\xi]+2\cos[4\xi]+\cos[5\xi])\sigma_5^2\sigma_6^2\sigma_7 + \cos[\xi]\sigma_5^3\sigma_7^2$$

$$+(1+2\cos[2\xi])\sigma_3\sigma_6^2\sigma_7^2-\cos[\xi]\sigma_5(\sigma_6^4+4\cos[\xi]^2\sigma_3\sigma_7^3))+\sigma_2(2\sigma_3^2\sigma_7((13+18\cos[\xi]+19\cos[2\xi]$$
$$+10\cos[3\xi]+7\cos[4\xi]+2\cos[5\xi]+\cos[6\xi])\sigma_6^3+(2+\cos[2\xi])\sigma_5\sigma_6\sigma_7-2\cos[\xi]^2\sigma_4\sigma_7^2)+\sigma_7(-\sigma_4^3\sigma_6^2$$
$$+(31+8\cos[\xi]+52\cos[2\xi]+12\cos[3\xi]+32\cos[4\xi]+12\cos[5\xi]+12\cos[6\xi]+4\cos[7\xi]$$
$$+2\cos[8\xi])\sigma_4\sigma_6\sigma_7^2-2((6+7\cos[2\xi]+2\cos[4\xi])\sigma_6^4+8\cos[\xi]^3(2\cos[\xi]-2\cos[2\xi]$$
$$+3\cos[3\xi]+2\cos[4\xi]+\cos[5\xi])\sigma_5\sigma_6^2\sigma_7-2\cos[\xi]^2\sigma_5^2\sigma_7^2))+\sigma_3(2(13+18\cos[\xi]$$
$$+19\cos[2\xi]+10\cos[3\xi]+7\cos[4\xi]+2\cos[5\xi]+\cos[6\xi])\sigma_4^2\sigma_6\sigma_7^2-\sigma_4(\sigma_6^4+2(18+22\cos[\xi]$$
$$+25\cos[2\xi]+12\cos[3\xi]+8\cos[4\xi]+2\cos[5\xi]+\cos[6\xi])\sigma_5\sigma_6^2\sigma_7-\sigma_5^2\sigma_7^2)-\sigma_7(\sigma_5^3\sigma_6$$
$$+8\cos[\xi]^2\cos[2\xi]\sigma_7^3)))))) \tag{D.6}$$

E. Minimal models

In this appendix we explicitly derive the form factor equations which characterize the operator content of the $\Phi_{1,3}$-perturbed minimal model $\mathcal{M}_{2,2N+3}$. The particles in such models obey the following bootstrap fusion algebra

$$B_a \times B_b \to B_{\min(a+b,2N+1-a-b)} \text{ and } B_a \times B_b \to B_{|a-b|}\,, \qquad a,b=1,\dots,N \tag{E.1}$$

which correspond to the bound state poles located at

$$\theta_{ab}^{\min(a+b,2N+1-a-b)} = i(a+b)\frac{\xi_N}{2} \tag{E.2}$$

$$\theta_{ab}^{|a-b|} = i(\pi-|a-b|\frac{\xi_N}{2})\,,\ \ a\neq b\,. \tag{E.3}$$

The pattern (E.1) makes clear the possibility to describe all the particles and their fusions in terms of the fusion processes involving only the lightest particle B_1. In this way all the bound state constraints can be handled inside the parametrization (3.7).

In particular, we can describe the particle $B_{\min(n,2N+1-n)}$ by the fusion of n B_1 particles set in pairs on the bound state pole of the B_1B_1 scattering channel. For the form factors of the generic local operator Φ this implies

$$\langle 0|\Phi(0)|B_{\min(n,2N+1-n)}(\tilde\theta_n)B_1(\theta_1')\dots B_1(\theta_m')\rangle\ =$$
$$\frac{(-i)^{n-1}}{\Upsilon_n}\lim_{\epsilon_1\to 0}\dots\lim_{\epsilon_{n-1}\to 0}\epsilon_1\cdot\cdot\epsilon_{n-1}F^{\Phi}_{m+n}(\theta_1+\epsilon_1,..,\theta_{n-1}+\epsilon_{n-1},\theta_n,\theta_1',..,\theta_m') \tag{E.4}$$

for $1\le n\le 2N$, $\theta_h=\theta_1+i(h-1)\xi_N$, $h=2,\dots,n$, and $\tilde\theta_n=\theta_1+i(n-1)\xi_N/2$. The result does not depend on the way in which the limit is done and Υ_n can be written as

$$\Upsilon_n=\begin{cases}\prod_{k=1}^{n}\Gamma_{1k}^{k+1} & \text{for } 1\le n\le N\,,\\ \prod_{k=1}^{N}\Gamma_{1k}^{k+1}\prod_{h=2N+2-n}^{N}\Gamma_{1h}^{h-1} & \text{for } N<n\le 2N,\end{cases}\,, \tag{E.5}$$

where Γ_{ab}^c is the three-particle coupling defined by

$$\mathrm{Res}_{\theta=\theta_{ab}^c}S_{ab}(\theta)=i\,(\Gamma_{ab}^c)^2\,. \tag{E.6}$$

The use of formula (E.4) allows to express the conditions (4.22) as

$$\lim_{\epsilon_1\to 0}...\lim_{\epsilon_{2N-n}\to 0}\epsilon_1\cdot\cdot\epsilon_{2N-n}F^{\Phi}_{2N+1-n}(\theta'_1+\epsilon_1,..,\theta'_{2N-n}+\epsilon_{2N-n},\theta'_{2N+1-n}) =$$
$$(-1)^{N-n}\frac{\Upsilon_n}{\Upsilon_{2N+1-n}}\lim_{\epsilon_1\to 0}...\lim_{\epsilon_{n-1}\to 0}\epsilon_1\cdot\cdot\epsilon_{n-1}F^{\Phi}_n(\theta_1+\epsilon_1,..,\theta_{n-1}+\epsilon_{n-1},\theta_n)\ , \tag{E.7}$$

with θ_h defined as above, $1\leq n\leq N$, $\theta'_h=\theta'_1+i(h-1)\xi_N$, $h=2,\ldots,2N+1-n$, and $\theta'_1=\theta_1-i(2(N-n)+1)\xi_N/2$. Moreover, for each $m+2N$ B_1 particles with $m\geq 0$, we can consider the case $n=2N$ in (E.4) which implies the equation [36]

$$\lim_{\epsilon_1\to 0,..,\epsilon_{2N-1}\to 0}\epsilon_1\cdot\cdot\epsilon_{2N-1}F^{\Phi}_{m+2N}(\theta+\frac{2N-1}{2}\xi_N+\epsilon_1,..,\theta-\frac{2N-1}{2}\xi_N,\theta_1,..,\theta_m) =$$
$$i^{(2N-1)}\Upsilon_{2N}F^{\Phi}_{m+1}(\theta,\theta_1,..,\theta_m)\ . \tag{E.8}$$

In terms of the parametrization (3.7) the above equations become

$$Q^{\Phi}_{2N+1-n}(\theta'_1,..,\theta'_{2N+1-n})=W^N_n(\theta_1)Q^{\Phi}_n(\theta_1,..,\theta_n)\ , \tag{E.9}$$

$$W^N_n(\theta_1)=\frac{Q^{(1)}_{2N+1-n}(\theta'_1,..,\theta'_{2N+1-n})}{Q^{(1)}_n(\theta_1,\ldots,\theta_n)} \tag{E.10}$$

and

$$Q^{\Phi}_{m+2N}(\theta+\frac{2N-1}{2}\xi_N,\ldots,\theta+\frac{1}{2}\xi_N,\theta-\frac{1}{2}\xi_N,\ldots,\theta-\frac{2N-1}{2}\xi_N,\theta_1,\ldots,\theta_m) =$$
$$V^N_m(x,x_1,\ldots,x_m)Q^{\Phi}_{m+1}(\theta,\theta_1,\ldots,\theta_m)\ , \tag{E.11}$$

with

$$V^N_m(x,x_1,\ldots,x_m)=(-1)^{\frac{N(N+1)}{2}+1}\prod_{k=2}^{N-1}[k]^2x^{N(2N-1)}\prod_{i=1}^{m}(x+x_i)\prod_{k=2}^{N}\prod_{i=1}^{m}(x-x_i\omega^k)(x-x_i\omega^{-k})\ , \tag{E.12}$$

$x_i=e^{i\theta_i}$ and $\omega=e^{i\xi_N}$.

These bound state equations complete the equations (3.18)-(3.21) and characterize the operator content of this model. Let us describe the case of the primary fields. For $\xi_N=2\pi/(2N+1)$ a periodicity arises and the set of exponential operators $e^{ik\beta\varphi}$ of the sine-Gordon model reduces to a finite set of $2N$ independent ones. Indeed, the form factors of $e^{ik\beta\varphi}$ satisfy

$$F^{k+(2N+1)}_n(\theta_1,..,\theta_n)=F^k_n(\theta_1,..,\theta_n)\ , \tag{E.13}$$

$$F^{(2N+1)-k}_n(\theta_1,..,\theta_n)=(-1)^{n+1}F^k_n(\theta_1,..,\theta_n)\ , \tag{E.14}$$

implying that the independent among the operators $e^{ik\beta\varphi}$ can be chosen for $k=1,..,2N$. Now, only those with $k=1,..,N$ have form factors satisfying the bound state equations (E.9) and (E.11). Thus, the reduction of the operator content of the $\Phi_{1,3}$-perturbed minimal model $\mathcal{M}_{2,2N+3}$ is implemented simply requiring the bound state equations.

The form factors of the operator $T\bar{T}$ that we computed explicitly up to nine particles satisfy the bound state equations (E.9) and (E.11) for any N when setting $\xi=\xi_N$, as required by the property ii) in section 4.

F. Free limit

The expressions given in this paper for the form factors of the components of the energy-momentum tensor and for $T\bar{T}$ apply to the sinh-Gordon model seen as a perturbation of the Gaussian (C=1) fixed point when the form factors on states containing an odd number of particles are set to zero. In this case the limit $b = 0$ corresponds to a free massive boson with background charge $Q = 0$. The form factors of Θ and $T\bar{T}$ behave as

$$F^{\Theta}_{2m} \sim b^{2(m-1)}\,, \qquad F^{T\bar{T}}_{2m} \sim b^{2(m-2)} \tag{F.1}$$

as $b \to 0$. As expected, the only finite non-vanishing form factors at $b = 0$ are obtained for 2 and 4 particles, respectively; there are however infinities at 0 particles for Θ and at 0 and 2 particles for $T\bar{T}$ that need to be subtracted.

The usual finite trace operator with zero expectation value at $b = 0$ is defined through the subtraction[17]

$$\Theta_R = \Theta - \langle\Theta\rangle\, I\ . \tag{F.2}$$

The divergences of $T\bar{T}$ are eliminated in the subtracted expression

$$\begin{aligned} T\bar{T}_R &= T\bar{T} - a\, m^{-2}\, \partial^2\bar{\partial}^2\Theta - d\, m^2\, \Theta - e\, m^4\, I + \mathcal{F} \\ &= \mathcal{K} + c\, \partial\bar{\partial}\Theta + \mathcal{F}\,, \end{aligned} \tag{F.3}$$

where a, d and e are the (infinite at $b = 0$) coefficients given in section 4, and $\mathcal{F}$ is a finite part needed to ensure that the asymptotic factorization (4.6) continue to hold for the regularized operators after the subtraction of the term proportional to $\partial^2\bar{\partial}^2\Theta$. This implies that $F^{\mathcal{F}}_n$ vanishes for $n \neq 4$ and is determined by

$$Q^{\mathcal{F}}_4 = \langle\Theta\rangle^2\, \frac{\sigma_1^2\sigma_3^2}{\sigma_4^2}\, (\sigma_1\sigma_3\sigma_2 - \sigma_3^2 - \sigma_1^2\sigma_4)\,, \tag{F.4}$$

so that

$$F^{T\bar{T}_R}_4 = \left(\frac{\pi m^2}{2}\right)^2 \frac{1}{\sigma_4}(\sigma_2^2 + 14\sigma_1\sigma_3 - 4\sigma_4)\ . \tag{F.5}$$

If $F^{T\bar{T}_R}_2$ is set to zero choosing $c = 0$, this is the only non-vanishing form factor of $T\bar{T}_R$.

This result is easily compared with that for the matrix elements of $T(x)\bar{T}(x) - \Theta(x)\Theta(x)$ computed in free field theory with normal ordering regularization. The only difference arises in the coefficient of the term proportional to $\sigma_1\sigma_3$ in (F.5), amounting to a contribution of the operator $\partial\bar{\partial}\phi^4$. This is precisely one of the derivative terms in (4.3) left unfixed by (4.6).

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